Appendix B. Skipped Cases

The script **ListGroup** returns 3 output: the first is processed by the other scripts that possibly return some surfaces. The other output are not touched by the other scripts. We will show that all these cases do not occur.

One of the main tools here is the script Test (or TestBAD in some cases), which checks, given a signature and an order, if there exist groups with that order and with a spherical system of generators of that signature.

For all the value $1 \le K_S^2 \le 8$, we have that the second output is empty, while the cases stored in the third output are collected in the following table:

K_S^2	Sing$X$	type	$\lvert G^0 \rvert$
4	$4 \times A_1$	2, 3, 8	2304
5	$3 \times A_1$	2, 3, 8	2880
6	$2 \times A_1$	2, 4, 5	2400
6	$2 \times A_1$	2, 3, 8	3456
7	$1 \times A_1$	2, 3, 9	2268
7	$1 \times A_1$	2, 4, 5	2800
7	$1 \times A_1$	2, 3, 8	4032
8	$\emptyset$	2, 3, 9	2592
8	$\emptyset$	2, 4, 5	3200
8	$\emptyset$	2, 3, 8	4608

Table 1. The skipped cases

In the following we sometimes affirm that there are no perfect group of the given order, this control is done using the following MAGMA function:

`NumberOfGroups(PerfectGroupDatabase(),order);`

while the other functions that we use are taken from the MAGMA script that we have developed.

B.0.1. *$K^2 = 4$, $K^2 = 5$ and $K^2 = 6$.*

Lemma B.1. *No group of order 2304, 2880 or 3456 has a spherical system of generators of type* $[2, 3, 8]$.

Proof. Assume that G^0 is a group of order 2304 (2880, 3456 resp.) admitting an appropriate homomorphism $\mathbb{T}(2, 3, 8) \to G^0$.

Since $\mathbb{T}(2, 3, 8)^{\mathrm{ab}} \cong \mathbb{Z}_2$ and since there are no perfect groups of order 2304 (2880, 3456 resp.), the commutator subgroup $G^{0\prime} = [G^0, G^0]$ of G^0 has order 1152 (1440, 1728 resp.). It easy to see that $[\mathbb{T}(2, 3, 8), \mathbb{T}(2, 3, 8)] \cong \mathbb{T}(3, 3, 4)$ and $G^{0\prime}$ has a spherical system of generators of type $[3, 3, 4]$, The following MAGMA computations

```
> TestBAD([3,3,4], 1152);
{}
```

```
>
> Test([3,3,4], 1440);
{}
>
> Test([3,3,4], 1728);
{}
>
```

show that there are no groups of order 1152 (1440, 1728 resp.) with a spherical system of generators of type $[3, 3, 4]$, a contradiction. □

Lemma B.2. *No group of order 2400 has a spherical system of generators of type* $[2, 4, 5]$.

Proof. Assume that G^0 is a group of order 2400 admitting an appropriate homomorphism $\mathbb{T}(2, 4, 5) \to G^0$.

Since $\mathbb{T}(2, 4, 5)^{\mathrm{ab}} \cong \mathbb{Z}_2$ and since there are no perfect groups of order 2400, the commutator subgroup $G^{0\prime} = [G^0, G^0]$ of G^0 has order 1200. It easy to see that $[\mathbb{T}(2, 4, 5), \mathbb{T}(2, 4, 5)] \cong \mathbb{T}(2, 5, 5)$ and $G^{0\prime}$ has a spherical system of generators of type $[2, 5, 5]$. The following MAGMA computation

```
> Test([2,5,5], 1200);
{}
>
```

shows, that there are no groups of order 1200 with a spherical system of generators of type $[2, 5, 5]$, a contradiction. □

B.0.2. $K^2 = 7$.

Lemma B.3. *No group of order 2268 has a spherical system of generators of type* $[2, 3, 9]$.

Proof. Assume that G^0 is a group of order 2268 admitting an appropriate homomorphism $\mathbb{T}(2, 3, 9) \to G^0$.

Since $\mathbb{T}(2, 3, 9)^{\mathrm{ab}} \cong \mathbb{Z}_3$ and since there are no perfect groups of order 2268, the commutator subgroup $G^{0\prime} = [G^0, G^0]$ of G^0 has order 756. It holds $[\mathbb{T}(2, 3, 9), \mathbb{T}(2, 3, 9] \cong \mathbb{T}(2, 2, 2, 3)$ and $G^{0\prime}$ has a spherical system of generators of type $[2, 2, 2, 3]$. The following MAGMA computation

```
> Test([2,2,2,3], 756);
{}
>
```

shows that there are no groups of order 756 with a spherical system of generators of type $[2, 2, 2, 3]$, a contradiction. □

Lemma B.4. *No group of order 2800 has a spherical system of generators of type* $[2, 4, 5]$.

Proof. The proof is exactly the same of Lemma B.2 (there are no perfect groups of order 2800), with the use of the MAGMA script:

```
> Test([2,5,5], 1400);
{}
>
```

□

Lemma B.5. *No group of order 4032 has a pair of spherical system of generators of type* $[2,3,8]$ *which give the expected singularities, i.e. 2 nodes.*

Proof. Assume that G^0 is a group of order 4032 admitting an appropriate homomorphism $\mathbb{T}(2,3,8) \to G^0$.

Since $\mathbb{T}(2,3,8)^{\mathrm{ab}} \cong \mathbb{Z}_2$ and since there are no perfect groups of order 4032, the commutator subgroup $G^{0'} = [G^0, G^0]$ of G^0 has order 2016. It is a quotient of $[\mathbb{T}(2,3,8), \mathbb{T}(2,3,8] \cong \mathbb{T}(3,3,4)$ and $G^{0'}$ has a spherical system of generators of type $[3,3,4]$.

Since $\mathbb{T}(3,3,4)^{\mathrm{ab}} \cong \mathbb{Z}_3$ and since there are no perfect groups of order 2016, the commutator subgroup $G^{0''} = [G^{0'}, G^{0'}]$ of $G^{0'}$ has order 672; it is a quotient of $[\mathbb{T}(3,3,4), \mathbb{T}(3,3,4)] \cong \mathbb{T}(4,4,4)$ and $G^{0''}$ has a spherical system of generators of type $[4,4,4]$. The following MAGMA computation

```
> Test([4,4,4], 672);
{ 1046, 1255 }
>
```

shows that only the groups $Gs(672, v)$ with $v \in \{1046, 1255\}$ have a spherical system of generators of type $[4,4,4]$.

We want to exclude these two cases. Assume that $G^{0'}$ has a spherical system of generators of type $(3,3,4)$. Let consider the following commutative diagram:

$$\begin{tikzcd}
 & & \mathbb{T}(3,3,4) \arrow[r, "q"] \arrow[d, "p"] & \mathbb{T}(3,3,4)^{ab} = \mathbb{Z}_3 \arrow[d, equal] & \\
1 \arrow[r] & G^{0''} \arrow[r] & G^{0'} \arrow[r, "f"] & \mathbb{Z}_3 \arrow[r] & 1
\end{tikzcd}$$

where $q(c_i) = d_i$. Let

$$\mathbb{T}(3,3,4) = \langle c_1, c_2, c_3 \mid c_1^3, c_2^3, c_3^4, c_1c_2c_3 \rangle$$

$$\begin{aligned}
\mathbb{T}(3,3,4)^{ab} &= \langle d_1, d_2, d_3 \mid d_1^3, d_2^3, d_3^4, d_1d_2d_3, [d_i, d_j]_{1 \le i,j \le 3} \rangle \\
&= (\mathbb{Z}_3 d_1 \times \mathbb{Z}_3 d_2 \times \mathbb{Z}_4 d_3)/\langle (1,1,1) \rangle
\end{aligned}$$

since $[d_1] = (1,0,0) \notin \langle (1,1,1) \rangle$, then $[d_1] \neq [0]$; so we have $q(c_1) \neq [0]$, and $f(p(c_1)) = \pi(g_1) \neq 0$. We have found an element of $G^{0'}$ of order 3 that does not belong to $G^{0''}$, this means that the following exact sequence

$$1 \longrightarrow G^{0''} \longrightarrow G^{0'} \xrightarrow{f} \mathbb{Z}_3 \longrightarrow 1$$

splits with map

$$\begin{aligned}
\alpha : \mathbb{Z}_3 &\longrightarrow G^{0'} \\
d_1 &\longmapsto g_1
\end{aligned}$$

and so $G^{0'} \cong G^{0''} \rtimes \mathbb{Z}_3$.

The next claim, that we not prove, is a standard result about semidirect product.

Claim. Let L be a finite group and let K be a cyclic group of order p. Let $\varphi_1, \varphi_2 : K \to \mathrm{Aut}(L)$ such that $\varphi_1(K)$ and $\varphi_2(K)$ are conjugated. Then $L \rtimes_{\varphi_1} K \cong L \rtimes_{\varphi_2} K$.

This means that, in order to build up the group $G^{0'}$, we have only to look at the conjugacy classes of elements of order 3 in $\mathrm{Aut}(G^{0''})$ and at $\mathrm{Id}(\mathrm{Aut}(G^{0''}))$.

The following MAGMA script shows that $G^{0''} = G(672, 1046)$ has 56 automorphisms of order 3 included in the same conjugacy class; hence, up to isomorphisms, there are at most two $G^{0''} \rtimes \mathbb{Z}_3$, one given by a representative of that conjugacy class and the other is the direct product! The script shows also that these two extensions $G^{0''} \rtimes \mathbb{Z}_3$ do not have a spherical system of generators of type $(3, 3, 4)$, hence these cases do not occur.

```
> H2:=SmallGroup(672,1046);
> R2:=AutConjugCl(H2,3);
56
1
> C3:=CyclicGroup(3);
> Aut2:=AutomorphismGroup(H2);
> R2[2]:=Id(Aut2);
> f:=[]; for i in [1..2] do f[i]:=hom<C3->Aut2|R2[i]>;end for;
h1:=[]; for i in [1..2] do h1[i]:=SemidirectProduct(H2,C3,f[i]);
for> i, ExSphGens(h1[i],[3,3,4]); end for;
1 false
2 false
>
```

The following MAGMA script shows that $G^{0''} = Gs(672, 1255)$ has 170 automorphisms of order 3 divided in three conjugacy classes; hence, up to isomorphisms, there are at most four $G^{0''} \rtimes \mathbb{Z}_3$, the additional one is given by the direct product. It also shows that two of these four extensions $G^{0''} \rtimes \mathbb{Z}_3$ do not have a spherical system of generators of type $(3, 3, 4)$, while the other two have a spherical system of generators of type $(3, 3, 4)$. Moreover this two extensions are isomorphic.

```
> H2:=SmallGroup(672,1255);
> R2:=AutConjugCl(H2,3);
170
3
> C3:=CyclicGroup(3);
> Aut2:=AutomorphismGroup(H2);
> R2[4]:=Id(Aut2);
> f:=[]; for i in [1..4] do f[i]:=hom<C3->Aut2|R2[i]>;end for;
> h1:=[]; for i in [1..4] do h1[i]:=SemidirectProduct(H2,C3,f[i]);
for> i, ExSphGens(h1[i],[3,3,4]); end for;
1 true
```

```
2 false
3 true
4 false
> IsIsomorphic(h1[1],h1[3]);
true  Homomorphism of ...
>H1:=h1[1];
```

It can be proved, in a similar way as for $G^{0'} \cong G^{0''} \rtimes \mathbb{Z}_3$, that G^0 is isomorphic to a semidirect product $G^{0'} \rtimes \mathbb{Z}_2$.

The following MAGMA script (that continues the previous one) shows that $G^{0'}$=h1[1] has eight conjugacy classes of automorphisms of order 2; so, up to isomorphisms, there are at most nine $G^{0'} \rtimes \mathbb{Z}_2$ (one is given by the direct product). It also shows that eight of these nine extensions $G^{0'} \rtimes \mathbb{Z}_2$ do not have a spherical system of generators of type $(2,3,8)$, while the other one does.

```
> R1:=AutConjugCl(H1,2);
499
8
>C2:=CyclicGroup(2);
> Aut1:=AutomorphismGroup(H1);
> R1[9]:=Id(Aut1);
>f:=[]; for i in [1..9] do f[i]:=hom<C2->Aut1|R1[i]>;end for;
> h:=[]; for i in [1..9] do h[i]:=SemidirectProduct(H1,C2,f[i]);
for> i, ExSphGens(h[i],[2,3,8]); end for;
1 true
2 false
3 false
4 false
5 false
6 false
7 false
8 false
9 false
>
```

The following MAGMA script shows that for each pair of spherical systems of generators of type $[2,3,8]$ of G^0=h[1], the singularities test fails, and so also this case does not occur.

```
>H:=h[1];
> SingularitiesY([0,1],[2,3,8],H);
false
```

$\square$

B.0.3. $K^2 = 8$. We recall that a pair of spherical systems generators (T_1, T_2) is *disjoint* if

$$\Sigma(T_1) \cap \Sigma(T_2) = \{1\}\,.$$

If (a_1, b_1, c_1) and (a_2, b_2, c_2) are a disjoint pair of spherical system of generators of type $[2,3,9]$ for G^0, then $(a_i, b_i a_i b_i^{-1}, b_i^2 a_i b_i^{-2}, c_i^3)$, for $i = 1, 2$, are spherical system of generators of type $[2,2,2,3]$ for $G^{0\prime} = [G^0, G^0]$; moreover these two systems are disjoint (see [BCG08, Lemma 4.3]).

Lemma B.6. *No group of order 2592 has a disjoint pair of spherical system of generators of type* $[2,3,9]$*, that is* $Y = (C \times C)/G^0$ *is smooth.*

Proof. Assume that G^0 is a group of order 2592 admitting an appropriate homomorphism $\mathbb{T}(2,3,9) \to G^0$.

Since $\mathbb{T}(2,3,9)^{\mathrm{ab}} \cong \mathbb{Z}_3$ and since there are no perfect groups of order 2592, the commutator subgroup $G^{0\prime} = [G^0, G^0]$ of G^0 has order 864, it is a quotient of $[\mathbb{T}(2,3,9), \mathbb{T}(2,3,9] \cong \mathbb{T}(2,2,2,3)$ and $G^{0\prime}$ has a spherical system of generators of type $[2,2,2,3]$. The following MAGMA computation

```
> Test([2,2,2,3], 864);
{2225, 4175}
>
```

shows that only the groups $Gs(864, v)$ with $v \in \{2225, 4175\}$ have a spherical system of generators of type $[2,2,2,3]$.

We recall that a pair of spherical systems generators (T_1, T_2) is *disjoint* if

$$\Sigma(T_1) \cap \Sigma(T_2) = \{1\}.$$

If (a_1, b_1, c_1) and (a_2, b_2, c_2) are a disjoint pair of spherical system of generators of type $[2,3,9]$ for G^0, then $(a_i, b_i a_i b_i^{-1}, b_i^2 a_i b_i^{-2}, c_i^3)$, for $i = 1, 2$, are spherical system of generators of type $[2,2,2,3]$ for $G^{0\prime} = [G^0, G^0]$; moreover these two systems are disjoint.

The following MAGMA computations

```
> SingularitiesY([0,0],[2,2,2,3],SmallGroup(864,2225));
false
> SingularitiesY([0,0],[2,2,2,3],SmallGroup(864,4175));
false
```

show that the groups $Gs(864, 2225)$ and $Gs(864, 4175)$ do not have a disjoint pair of spherical system of generators of type $[2,2,2,3]$, a contradiction. □

Lemma B.7. *No group of order 3200 has a spherical system of generators of type* $[2,4,5]$.

Proof. The proof is exactly the same of Lemma B.2 (there are no perfect groups of order 2), with the use of the MAGMA script:

```
> Test([2,5,5], 1600);
{}
>
```

□

Lemma B.8. *No group of order 4608 has a spherical system of generators of type* $[2,3,8]$.

Proof. Assume that G^0 is a group of order 4608 admitting an appropriate homomorphism $\mathbb{T}(2,3,8) \to G^0$.

Since $\mathbb{T}(2,3,8)^{\mathrm{ab}} \cong \mathbb{Z}_2$ and since there are no perfect groups of order 4608, the commutator subgroup $G^{0\prime} = [G^0, G^0]$ of G^0 has order 2304, it is a quotient of $[\mathbb{T}(2,3,8), \mathbb{T}(2,3,8] \cong \mathbb{T}(3,3,4)$ and $G^{0\prime}$ has a spherical system of generators of type $[3,3,4]$.

Since $\mathbb{T}(3,3,4)^{\mathrm{ab}} \cong \mathbb{Z}_3$ and since there are no perfect groups of order 2304, the commutator subgroup $G^{0\prime\prime} = [G^{0\prime}, G^{0\prime}]$ of $G^{0\prime}$ has order 768, it is a quotient of $[\mathbb{T}(3,3,4), \mathbb{T}(3,3,4)] \cong \mathbb{T}(4,4,4)$ and $G^{0\prime\prime}$ has a spherical system of generators of type $[4,4,4]$. The following MAGMA computation

```
> TestBAD([4,4,4], 768);
{}
>
```

shows that there are no groups of order 768 with a spherical system of generators of type $[4,4,4]$, a contradiction. $\square$